Dynamics of kicked particles in a double-barrier structure

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We study the classical and quantum dynamics of periodically kicked particles placed initially within an open, double barrier structure. This system does not obey the Kolmogorov-Arnold-Moser (KAM) theorem and displays chaotic dynamics. The phase space features induced by non-KAM nature of the system leads to dynamical features such as the non-equilibrium steady state, classically induced saturation of energy growth and momentum filtering. We also comment on the experimental feasibility of this system as well as its relevance in the context of current interest in classically induced localisation and chaotic ratchets.

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I. INTRODUCTION

Periodically kicked rotor is a popular model that has served as a paradigm to understand Hamiltonian chaos both in the classical and quantum regime [1]. This was originally introduced as a simple model for dynamical chaos but was sufficiently general enough to cover many physical situations. For instance, problems like the Hydrogen atom in microwave fields and motion of a comet around the sun driven by a suitable planet can be reduced to that of kicked rotor [2]. This system is also paradigmatic for another important reason; it obeys KAM (Kolmogorov- Arnold-Moser) theorem [3]. This implies that, as a control parameter is varied, the transition from regularity to chaos occurs progressively by breaking of invariant curves in phase space. Once all the invariant curves are broken down, diffusive global transport of particles in phase space becomes possible. In the corresponding quantum regime, this classical diffusive transport is inhibited by the onset of dynamical localisation [4]. This was experimentally realised in the laboratory with cold atoms in optical lattices [5] and is the basis for theoretical and experimental realization of chaotic ratchets in recent times [6].

On the other hand, there are other physical systems that exhibit classical chaos but violate the KAM theorem, the so-called non-KAM systems. This class includes the kicked harmonic oscillator (KHO) [7] and the kicked particle in infinite square well potential [8, 10]. In both these cases, when a parameter is varied, the invariant curves are replaced by stochastic webs [7], an intricate chain of islands and globally connected channels, through which particle transport becomes possible. Non-KAM type is also relevant for an important class of physical systems, namely the dynamics of particles in quantum wells and barrier structures. Till date, non-KAM systems have been experimentally realised in semiconductor superlattices in tilted magnetic fields in which the enhanced conductivity could be attributed to non-KAM chaos [11]. Further, measurement of Lochschmidt echo using a non-KAM system, namely, the ion trap with harmonic potential in the presence of a kicking field [12] has also been proposed. Inspite of this, very few non-KAM systems have been investigated and they have not been explored in sufficient details.

Another motivation for this work stems from the considerable interest in recent times in the dynamics of condensates placed in finite box-type potentials acted upon by a periodically kicking field. In a recent experiment, Henderson et. al. [13] have constructed a quasi-1D finite box, using a combination of optical and magnetic trap, with the Bose-Einstein condensates in the box receiving periodic kicks. This set-up was used to study the effect of atomic interactions on the transport of BECs. In place of the dynamical localisation they observed a classical saturation in the energy of BECs due to a balance between the energy gained from kicks and the energy lost by leakage of BECs over the finite barrier [13]. Apart from this, a series of experiments [14] that studied the transport of BECs in the presence of disordered potential have reported such classically induced energy saturation effects. Then the question is if it is possible to observe such classically induced energy saturation in chaotic systems without inter-particle interactions and what would its mechanism be ? We show that kicked particle in finite well type potential that we study in this paper shows this feature and we discuss its mechanism. It is also relevant to point out that following the achievement of BECs in the optical box trap [15], theoretical investigations of resonance and anti-resonance behaviour and its relation to the KAM and non-KAM type dynamics for BECs in 1D infinite well have also been performed [16]. Further, experiments exploring the interface of nonlinear dynamics of electrons in 1D quantum well irradiated at terahertz frequencies have already been reported [17].

Although non-KAM type dynamics is a generic feature in physical systems such as the potential wells not much work has been done on this class of problems. However, on the theoretical front, infinite square well po-

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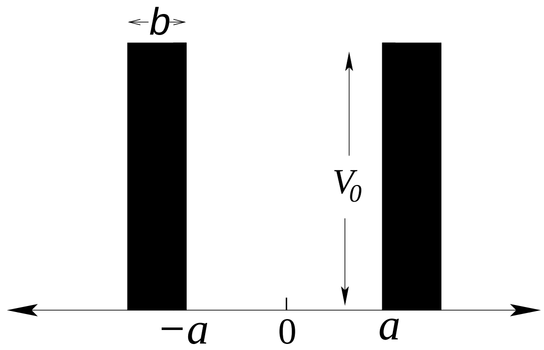

FIG. 1: Schematic of the stationary part of the potential. The width of each barrier is b. The well region has width $2a$.

tential confining a delta-kicked particle has been studied [8, 10]. In the light of recent attempts to study chaotic ratchets [6], in both the classical and quantum sense, it would be useful to open the potential to allow for particle transport. This could lead to chaotic ratchets that can utilise non-KAM features of its classical system for directed transport. In this paper, we study a periodically kicked particle initially held in between a finite double barrier structure. Double barrier heterostructures play an important role in electronic devices that use resonant tunnelling diodes [18] though without the kicking potential. Primarily we present numerical explorations of this problem to study its rich dynamical features. In the next section, we introduce our model and in subsequent sections we discuss the classical and quantum dynamics of this system.

II. THE KICKED PARTICLE IN DOUBLE BARRIER

We consider the dynamics of a non-interacting particle initially located in between two potential barriers each of height V_0 and width b distant $2a$ apart (see Fig. 1). The particle is further subjected to flashing δ-kicks of period T generated by a spatially periodic potential field of wavelength λ. Amplitude ϵ of the kicking field is generally referred to as kick strength. The classical Hamiltonian of the system is

$$\widetilde{H} = \frac{\widetilde{p}^2}{2m} + \widetilde{V}_{sq} + \widetilde{\epsilon}\cos\left(\frac{2\pi\widetilde{x}}{\lambda} + \phi\right)\sum_{n=-\infty}^{\infty}\delta(\widetilde{t} - nT), \quad (1)$$

where $\widetilde{V}_{sq} = \widetilde{V}_0\Big[\Theta(\widetilde{x}+\widetilde{b}+a) - \Theta(\widetilde{x}+a) + \Theta(\widetilde{x}-a) - \Theta(\widetilde{x}-a-\widetilde{b})\Big]$, $\Theta(.)$ is the unit step function and ϕ is the phase of the kicking field. The set of canonical transformations given by,

$$\widetilde{t} = tT, \quad \widetilde{x} = \lambda\frac{(x-\phi)}{2\pi}, \quad \widetilde{p} = \frac{pTE_c}{\lambda\pi},$$
$$\widetilde{H} = \frac{HE_c}{2\pi^2}, \quad \widetilde{\epsilon} = \frac{\epsilon E_c T}{2\pi^2}, \quad \widetilde{V}_0 = \frac{V_0 E_c}{2\pi^2}, \quad \widetilde{b} = \frac{\lambda}{2\pi}b, \quad (2)$$

with $E_c = m\lambda^2/2T^2$ leads to a new dimensionless Hamiltonian

$$H = \frac{p^2}{2} + V_{sq} + \epsilon\cos(x)\sum_{n=-\infty}^{\infty}\delta(t-n). \quad (3)$$

In this, $V_{sq} = V_0\left[\Theta(x-\phi+b+R\pi) - \Theta(x-\phi+R\pi) + \Theta(x-\phi-R\pi) - \Theta(x-\phi-R\pi-b)\right]$ with $R = 2a/\lambda$ being the ratio of the distance between the barriers to the wave length of the kicking field. The classical dynamics of the system depends upon five parameters, namely, ϵ, R, b, V_0 and ϕ. Of these, R, b and ϕ determine the positions of discontinuities in the potential (position of the wall boundaries) at $\mathbf{B} = \{-x_l - b, -x_l, x_r, x_r + b\}$ where $x_l = -R\pi + \phi$ and $x_r = R\pi + \phi$. Note that if $\phi = 0$, then $x_l = -x_w$ and $x_r = x_w$ with $x_w = R\pi$. Thus, the qualitative nature of the classical dynamics depends on the positions of wall boundaries collectively denoted by $\mathbf{B}$, the kick strength ϵ and potential height V_0. In this paper (except in section III(C)), we set $\phi = 0$ which makes the potential symmetric about $x = 0$. It is useful to write Eq. 3 as

$$H = H_0 + V_{sq}(x), \quad (4)$$

where $H_0 = \frac{p^2}{2} + \epsilon\cos(x)\sum_{n=-\infty}^{\infty}\delta(t-n)$ leads to standard map defined on the infinite plane. Note that if $V_{sq}(x) = V_0$, a constant, then the Hamilton's equations will not have the potential term and the dynamics would be completely governed by H_0.

III. CLASSICAL DYNAMICS

A. Classical map

The Hamiltonian in Eq. (3) is classically integrable for $\epsilon = 0$. This corresponds to free motion in the presence of two potential barriers and hence it is possible to obtain a transformation to action-angle variables. For $\epsilon > 0$, the system is non-integrable and can even display abrupt transition to chaotic dynamics with mixed phase space depending on the values of R, b and ϕ. It is convenient to think of the system as being entirely governed by H_0 and then incorporate effect of discontinuities in V_{sq} through appropriate boundary conditions. This leads to the following map,

$$\begin{aligned} p_n &= p_{n-1} + \epsilon\sin(x_{n-1}), \\ x_n &= x_{n-1} + p_n, \end{aligned} \quad (5a)$$

$$\begin{pmatrix} p_n \\ x_n \end{pmatrix} \to \widehat{\mathcal{R}}\begin{pmatrix} p_n \\ x_n \end{pmatrix}. \quad (5b)$$

Equation 5a represents the effect of H_0 and is identical to the standard map. In Eq. 5b, the operator $\widehat{\mathcal{R}} = \widehat{\mathcal{R}}_k \ldots \widehat{\mathcal{R}}_2\widehat{\mathcal{R}}_1$ represents the effect due to k encounters of the particle, in between two kicks, with the discontinuities of V_{sq} at positions represented by $B_1, B_2, \ldots B_k$

respectively. Depending on the energy, each of these k encounters could either be a reflection (sign of momentum changes) or refraction (magnitude of momentum changes) at $B_i \in \mathbf{B}$, $i = 1, 2, ...k$.

The map in Eq. 5 would be complete if the operator $\widehat{\mathcal{R}}_i$, that incorporates effect of i^{th} discontinuity encountered, is explicitly written down. Between successive kicks applied at times n and $n+1$, we denote the state of the particle after incorporating effect of ith encounter with a boundary B_i by $\begin{pmatrix} x_n^i \\ p_n^i \end{pmatrix}$. We define $]x_s^i, x_n^i[$ with $i = 0, 1 \ldots k$ as the path, starting from x_s^i, a particle would traverse between the two kicks after encountering i^{th} discontinuity if there were no discontinuities to be faced till the next kick. For $i = 0$, x_s^i would simply be x_{n-1} and would be equal to B_i for $i > 0$. x_n^0 and p_n^0 to be used in boundary conditions would simply be x_n and p_n obtained directly from Eq. 5a. Boundary conditions defined by Eq. 6 below are applied k times until $]x_s^k, x_n^k[\, \cap \mathbf{B} = \emptyset$. If E_n denotes the energy of the system at nth kick, then for $E_n \leq V_0$ (reflective boundary condition), we obtain

$$\widehat{\mathcal{R}}_i \begin{pmatrix} x_n^{i-1} \\ \\ p_n^{i-1} \end{pmatrix} = \begin{pmatrix} 2B_i - x_n^{i-1} \\ \\ -p_n^{i-1} \end{pmatrix}. \tag{6a}$$

For $E_n > V_0$ (refractive boundary condition), we get

$$\widehat{\mathcal{R}}_i \begin{pmatrix} x_n^{i-1} \\ \\ p_n^{i-1} \end{pmatrix} = \begin{pmatrix} B_i + \dfrac{\left(x_n^{i-1} - B_i\right) p_n^i}{p_n^{i-1}} \\ \\ \dfrac{p_n^{i-1}}{|p_n^{i-1}|} \sqrt{\left(p_n^{i-1}\right)^2 - \dfrac{2V_0 V_{diff}}{|V_{diff}|}} \end{pmatrix}. \tag{6b}$$

In this, we have used $V_{diff} = V(x_n^{i-1}) - V(x_s^{i-1})$. Thus, the dynamics of system in Eq. 3 can be described by the standard map with $-\infty \leq x_n, p_n \leq \infty$ (Eq. 5) subjected to potential barriers (Eq. 5b). Notice that by putting $V_0 = 0$ in Eq. 6b, we obtain $\widehat{\mathcal{R}}_i = \mathbf{I}$ for all i, where $\mathbf{I}$ is the identity matrix of order 2. Then $\widehat{\mathcal{R}} = \mathbf{I}$ and, as expected, Eq. 5 reduces to standard map for $V_0 = 0$. Thus, the transformation (6) can be viewed as deviation from standard map dynamics induced after each encounter of the particle with the a discontinuity of potential V_{sq}.

B. Phase space features

Figure 2 shows a stroboscopic section obtained by evolving the map in Eq. 5 for uniformly distributed initial conditions in $x \in (-x_w, x_w), p \in (-p_c, p_c)$, where $p_c = \sqrt{2mV_0}$ is the minimum momentum required for barrier crossing. In this paper, we have chosen kick strength $\epsilon << 1$ such that the corresponding standard map displays only KAM curves. Firstly, a striking feature is the absence of invariant curves and the appearance of a mixed phase space. This is in stark contrast with the standard map which displays mostly quasi-periodic orbits for kick strengths of this order. This figure also shows snap shots (solid circles in red) of trajectory in-between successive encounters with the discontinuities at B. Clearly, the evolution between two successive encounters with the boundaries is confined to a trajectory that is identical with one of the quasiperiodic orbits of the corresponding standard map (obtained from Eq. 5 with $V_0 = 0$) shown as continuous lines in the figure. Due to V_{sq}, particle breaks away from one quasiperiodic orbit and joins another at each encounter with the boundaries. This leads to absence of quasiperiodic orbits and development of mixed phase space comprising intricate chains of islands embedded in chaotic sea. We illustrate the effects of discontinuities in Fig. 2 by following a typical initial condition marked A_1 in the chaotic layer. This evolves to A_2 on the invariant curve C_2 of the corresponding standard map. After a long time, this point appears on the curve C_1 and goes from A_3 to A_4. After a reflection at $-x_w$, it goes from A_5 to A_6 on C_3. Then it shifts to the barrier region $(x_w, x_w + b)$ and moves on C_7 from A_7 to A_8. Depending on the winding number of the orbit in $(x_w, x_w + b)$, the particle could have gone back in to region between the barriers or escape from the finite well. In the present example, it makes its escape out of two barrier structures and its state meets the curve C_5 at A_9. Once the particle has escaped, its state evolves

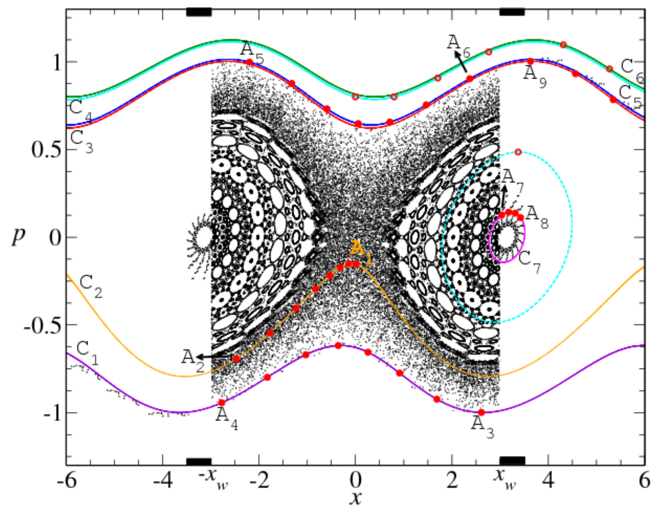

FIG. 2: (Color online) Stroboscopic Poincare section (black) for $R = 0.95, \epsilon = 0.15, V_0 = 0.5, \phi = 0$ and $b = 0.5$. All the continuous curves (in color) marked C_1 to C_6 are for the corresponding standard map with kick strength 0.15. The black box at position $x = \pm x_w$ indicates the width b of the barrier. The solid circles (in red) show a trajectory starting from A_1 until it exits the potential well at A_9. The time ordered sequence of the trajectory is A_1 to A_2, reflection at $-x_w$, A_3 to A_4, reflection at x_w, A_5 to A_6, cross the boundary at x_w, A_7 to A_8, cross the boundary at $x_w + b$, exit the potential at A_9. See text for details.

on same curve as $n \to \infty$. Thus, system displays KAM behaviour for $|x| > x_w + b$.

The absence of quasiperiodic orbits can be attributed to the non-analyticity of V_{sq} which violates the assumptions of KAM theorem. Thus, the non-KAM nature of system leads to onset of chaos even for $\epsilon < 1$. The initial conditions starting from chaotic layer will diffuse in momentum space. Some of these initial conditions which reach the set of quasiperiodic orbits $C(\mu)$ (μ being the winding number) of the corresponding standard map which overlaps the region $|p| > p_c$ can escape from the finite well. As μ increases, this overlap also increases and hence the escape probability is larger. This implies that there must exist μ_c such that the states on any $C(\mu)$, with $\mu > \mu_c$, will definitely cross the barrier and escape from the well. These orbits do not encounter the discontinuities in the potential multiple times and hence the energy of the particles evolving on such quasi-periodic orbits will not diffuse. Figure 2 also shows the trajectory of a particle (open circles in red on the curves C_5 and C_6) in such non-diffusive region. As seen in Fig. 2, the discontinuities at x_w and $x_w + b$ relocate the incoming particle from $C_5\,(\mu_5)$ to another orbit $C_6\,(\mu_6)$, where μ_5 and μ_6 are their winding numbers, respectively. As shown in Appendix B, when $b \to 0$, the effect of these discontinuities decreases and deviation between between two orbits measured as $(\mu_6 - \mu_5) \to 0$. This results in the appearance of regular orbits (see Fig. 3) identical to those of the standard map except that the former have imperceptible discontinuities wherever there is a discontinuity in potential. In other words, refraction becomes identity operation as $b \to 0$. Thus, the system shows regular dynamics outside region enclosed between curves $C_\pm\,(\mu_c)$ (see Fig. 3) as $b \to 0$. Note that the limits on chaotic phase space in terms of μ on positive and negative sides of momentum are identical due to assumption that $\phi = 0$. Limits on the chaotic phase space would exist even otherwise, though these would not be identical on both sides of $p = 0$. The discussions in this subsection can be summarized as follows ; we can define a phase space region $\mathcal{M}\,(|x| < x_w + b; |p\,(x)\,| < p\,(x; \mu_c))$, such that system has mixed phase space inside $\mathcal{M}$ in general and regular dynamics outside it. Here, $p\,(x; \mu_c)$ is momentum of any state on the curve $C_+\,(\mu_c)$ at position x. In Fig. 3, a close numerical approximation of the region $\mathcal{M}$ is highlighted by the red dashed line.

We remark that for $b \to 0$, the phase space structures inside $\mathcal{M}$ are identical to those of well map that describes the dynamics of δ-kicked particle in an infinite well [8]. This is to be expected since the well map has only reflective boundaries for $|p| \leq \infty$. Further, the well map is hyperbolic for $R < 0.5$ for any $\epsilon > 0$. The Hamiltonian in Eq. 1 also displays complete chaos for $R < 0.5$ inside $\mathcal{M}$. This is seen in Fig. 3 as no regular structures are visible in this region to the accuracy of our calculations. The region defined by $\mathcal{M}$ is determined by the positions of potential discontinuities $\mathbf{B}$ and $C_\pm(\mu_c)$. It can be shown that $C_\pm(\mu_c)$ will remain close to $\pm p_c (= \pm\sqrt{2mV_0})$ when

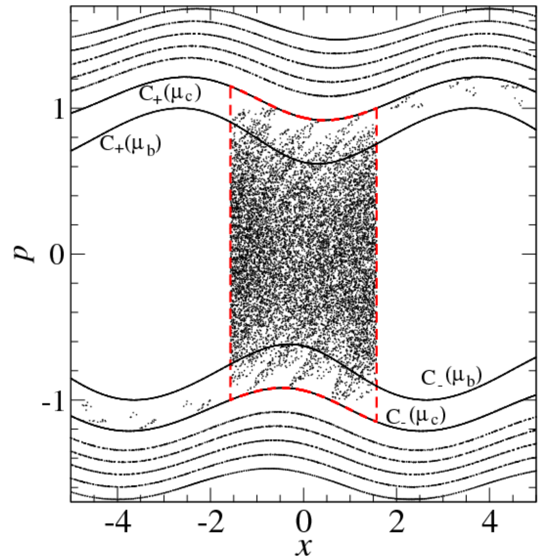

FIG. 3: Stroboscopic plot (excluding $C_\pm(\mu_b)$) for $b = 10^{-3}$ and $R = 0.5$. All the other parameters are same as in Figure 2. Dashed line (in red) represents the boundary of region $\mathcal{M}$. The mild scatter of points just below $C_+(\mu_c)$ and just above $C_-(\mu_c)$ represent the particles escaping out of the well (whose initial states were in $\mathcal{M}$). The curves $C_+(\mu_b)$ and $C_-(\mu_b)$ shown here are used in section V(B).

$b \to 0$ for any ϵ for which standard map has mostly regular phase space. Thus, the extent of chaotic region will depend grossly on the positions $\mathbf{B}$ and height V_0 of the barriers only. This implies that it is possible to engineer chaos in a desired region by varying these parameters.

C. KAM-like behavior: Role of symmetries

In this section, we explore the conditions under which KAM or non-KAM type of dynamics can be realized in the system. In Eq. 1, the non-analyticity of V_{sq} violates the assumptions of the KAM theorem. Hence, generically we expect this system to display the signatures of non-KAM system such as the stochastic webs instead of quasiperiodic orbits and an abrupt transition to chaos. These features are shown in Fig. 4(a,c,d,f). However, we show that even in the presence of non-analyticity in V_{sq}, quasiperiodic orbits similar to that in a KAM system can be realised if certain symmetry conditions are satisfied.

As argued before, until interrupted by the barriers, the dynamics is confined to a particular invariant curve of the corresponding standard map. We recall that corresponding to every trajectory C_+ of standard map with $p_n > 0$, there exists one and only one trajectory C_- with $p_n < 0$, such that a particle will evolve on these trajectories in exactly the same way but in opposite direction. As shown in appendix-A, consider the (R, ϕ) pairs for which the condition

$$\pm R\pi + \phi = l\ 2\pi, \qquad l \in \mathbb{Z} \tag{7}$$

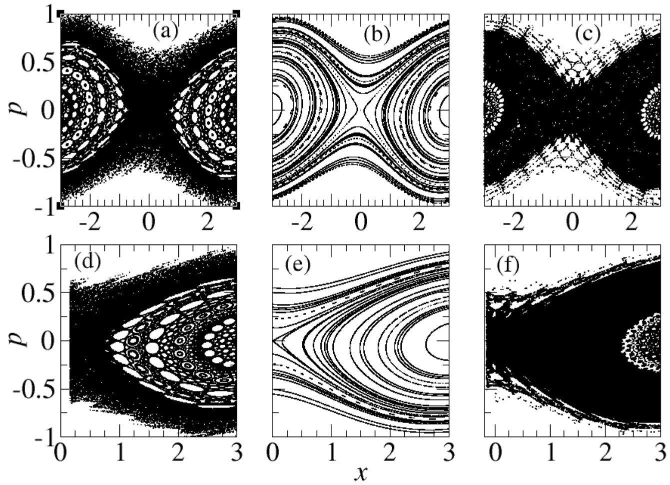

FIG. 4: Stroboscopic Poincare section for the Hamiltonian in Eq. 1 showing the region $x \in (-x_l, x_r), p \in (-p_c, p_c)$ for $b = 0, \epsilon = 0.15, V_0 = 0.5$. The other parameters are (a) $R = 0.95, \phi = 0$ (b) $R = 1.0, \phi = 0$, (c) $R = 1.05, \phi = 0$, (d) $R = 0.45, \phi = \pi/2$, (e) $R = 0.5, \phi = \pi/2$ and (f) $R = 0.55, \phi = \pi/2$.

is satisfied. When Eq. 7 is satisfied, application of $\widehat{\mathcal{R}}_i$ takes a particle from C_+ to C_- and application of $\widehat{\mathcal{R}}_{i+1}$ brings it back to C_+. This leads to quasiperiodic behavior in which the particle is confined to a pair of tori. This quasiperiodic orbit undergoes smooth deformation, just like in a KAM system, until it breaks for large kick strengths. Hence we call this KAM-like behaviour for its striking resemblance to the qualitative behaviour of a KAM system. In general, there exist infinite (R,ϕ) pairs for which KAM-like dynamical behaviour can be recovered in this system. In Fig 4(b,d), we show the sections for $R = 1, \phi = 0$ and $R = 0.5, \phi = \pi/2$ for which KAM-like behaviour is obtained. In Fig 4(a,c,d,f), we also show cases where Eq. 7 is not satisfied and hence for $|p| < p_c$ stochastic webs and chaotic regions are seen.

Symmetry related invariant curves like C_+ and C_- are due to the symmetry of the kicking field about any $x = m\pi + \phi$ where m is an integer. It turns out that when Eq. 7 is satisfied, kicking field is symmetric about x_w and x_{-w}. The existence of KAM-like behaviour in presence of non-analytic potential can be attributed to existence of centres of symmetry of kicking field at $-x_w$ and x_w.

IV. QUANTUM DYNAMICS

In this section, we discuss the quantum simulations of the system. We start by writing down the time-dependent Schroedinger equation corresponding to the scaled Hamiltonian in Eq. (3),

$$i\hbar_s \frac{\partial \psi}{\partial t} = \left[\frac{-\hbar_s^2}{2} \frac{\partial^2}{\partial x^2} + V_{sq} + \epsilon \cos x \sum_n \delta(t-n) \right] \psi. \quad (8)$$

The scaled Planck's constant is $\hbar_s = \frac{2\pi^2 \hbar}{E_c T}$. This being a kicked system, we can obtain the one-period Floquet operator,

$$\widehat{U} = \exp\left(-\frac{i\epsilon}{\hbar_s} \cos \widehat{x}\right) \exp\left(-\frac{i}{\hbar_s}\left[\frac{\widehat{p}^2}{2} + \widehat{V}_{sq}\right]\right), \quad (9)$$

such that $\psi(x, n) = \hat{U}^n \psi(x, 0)$. The classical limit will correspond to taking $\hbar_s \to 0$ keeping $\epsilon = \tilde{\epsilon}\hbar_s/\hbar$ constant. We calculate the Husimi distribution $Q(x_0, p_0, n)$ defined by

$$Q(x_0, p_0, n) = |\langle \psi(x, n)|x_0, p_0\rangle|^2 \quad (10)$$

for a wavepacket at time n. In this we take $\langle x|x_0, p_0\rangle$ as the minimum uncertainty wavepacket. In the semiclassical regime, the dynamics in the Husimi representation mimics the classical dynamics of the system in phase space [9]. In Fig. 5, we show the Husimi distribution at $n = 250$ from which one can clearly see that the density of Husimi distribution shows pattern similar to classical structures shown in Fig. 2.

Since $\widehat{p}$ and $\widehat{V}_{sq}$ in Eq. 9 do not commute, we first divide the duration between successive kicks into $N_{\triangle t}$ small time steps and the second term of Eq. (9) becomes $\prod_{i=1}^{N_{\triangle t}} \exp\left(-\frac{i}{\hbar_s N_{\triangle t}}\left[\frac{\widehat{p}^2}{2} + \widehat{V}_{sq}\right]\right)$. Then, we apply the split-operator method [19] to evolve the system. We use Fast Fourier transform [20] to obtain $\widetilde{\psi}(p)$ from $\psi(x)$ and vice-verse. In our calculations, we have taken $N_{\triangle t} \sim 2500$, the typical temporal step size is $O(10^{-3})$ and spatial step size is $O(10^{-4})$ to ensure that the evolved wavepackets converged to at least 8 decimal places.

The initial wavepacket at $n = 0$ is located in between the two barriers. We choose parameters b and $\hbar_s$ for which the Husimi distribution (shown in Fig. 5) closely resembles the classical phase space and shows that the probability density associated with the initial wavepacket will ultimately leave the barrier region by predominantly following the classical path rather than by tunnelling. Thus, the system stays in the semiclassical regime and tunnelling is suppressed. Quite clearly, for such a choice of parameters in the semiclassical regime, the classical dynamical features would be reflected in the quantum dynamics as well.

In the next two sections, we discuss some interesting dynamical features, namely (i) the non-equilibrium steady state (ii) classically induced suppression of diffusion and (iii) momentum filtering which primarily arise due to co-existence of diffusive (chaotic region ($\mu < \mu_c$)) region and non-diffusive region (regular region ($\mu > \mu_c$)) in same non-KAM system.

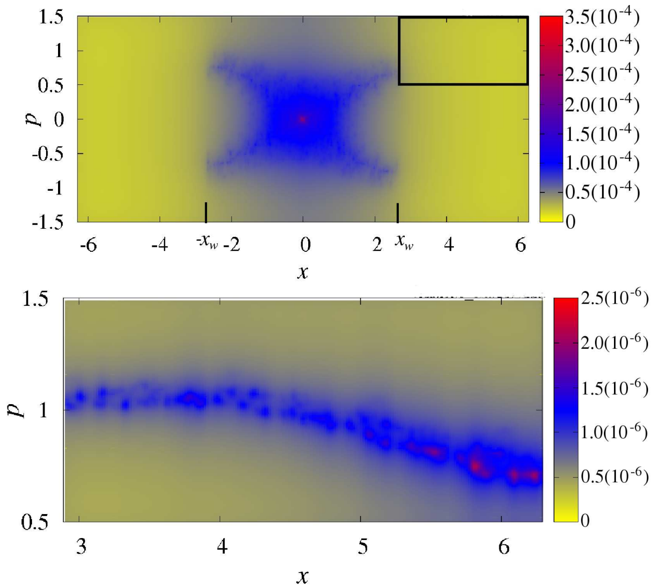

FIG. 5: (Color online) (Top) Husimi distribution for evolved wave packet. Initial wave function corresponds to $Q(x_0, p_0, n)$ sharply localised inside the chaotic region around $(0, 0)$. In the grey scale version, the regions with larger values of Husimi distribution function are grossly represented by the darker areas. It shows that the function decays very steeply outside $[x_{-w}, x_w]$ and acquires negligible values compared to those for region inside $[x_{-w}, x_w]$. We have taken $\hbar_s = 0.0025$, $R = 0.85$, $b = 0.2$, $\epsilon = 0.15$, $V_0 = 0.5$, $\phi = 0$. (Bottom) Enlarged and better resolved view of inset from figure on the top shows path followed by probability density outside the barrier region.

V. DYNAMICAL FEATURES

A. Non-equilibrium steady state

In this section, we show that the system in Eq. 1 can support non-equilibrium steady state (NESS) for intermediate time scales. We start with initial conditions uniformly distributed on a thin rectangular band around $p = 0$ stretched across the well region in between the potential barriers. As the kicking field begins to impart energy to the system, the particles which absorb sufficient energy escape from the well. At any time n, the mean energy $\langle E \rangle_{in}$ of the particles lying inside the well is $\left\langle \frac{p_n^2}{2} \right\rangle$, where $\langle . \rangle$ represents average at time n over the classical states (evolved from initial states over n kicking cycles) for which $-x_l < x < x_r$. In the corresponding quantum regime, we have,

$$\langle E \rangle_{in} = \int_{-x_l}^{x_r} \psi^*(x,n)\ \frac{\widehat{p}^2}{2} \psi(x,n)\ dx \tag{11}$$

The effect of the operator $\widehat{p}^2$ on $\psi(x, n)$ can be calculated using fast fourier transform and is equal to inverse fourier transform of $p^2 \widetilde{\psi}(p, n)$. Figure 6 shows that initially $\langle E \rangle_{in}$ increases and after a time scale t_r, $\langle E \rangle_{in}$ saturates to a constant. During this time scale, the behaviour is similar to the classical diffusive regime of the standard map.

The existence of steady state can be understood as follows. For the parameters used in Figure 6 the phase space in region $\mathcal{M}$ is fully chaotic. As kicks begin to act, any localized classical distribution $\rho_0(x, p)$ is quickly dispersed throughout this region. The total energy E_n of the particles in the well region increases. Simultaneously, the particles with $|p| > p_c$ leave the finite well leading to loss of energy. Soon the loss process becomes significant and at every kick cycle the energy lost (due to barrier crossings) is more than the energy gained from the kicking potential. Thus, E_n begins to decrease. However, after the time scale t_r, the net energy change and the number of particles vary in such a manner as to maintain the mean energy $\langle E \rangle_{in}$ a constant (apart from fluctuations). This arises because the normalised momentum distribution remains nearly invariant with time as shown in Fig. 7. The chaotic mixing inside the well ensures that, despite the loss of energetic particles, momentum distribution remain invariant. Thus, chaos between the barriers is essential to support the NESS. One of the factors that determine t_r is the rate at which any initial distribution of states diffuses in the chaotic region and steady state distribution shown in Fig. 7 is achieved. This rate increases with ϵ in general. For the present case with complete chaos, one expects this rate to be proportional to $1/\epsilon^2$, just like in the diffusive regime of standard map and hence one expects $t_r \propto 1/\epsilon^2$. Numerical results shown in Fig. 6 show a good agreement with this gross estimate for t_r.

This steady state holds good until nearly all the particles have escaped out and only a fraction $q << 1$ remains in the well. Based on rate of diffusion in chaotic region, we can estimate the time at which this happens to be $t_s \propto 1/q^2\epsilon^2$. Since $q << 1$, we get $t_s >> 1$. In the semiclassical regime, this mechanism carries over to the quantum dynamics as well. Notice that t_s is larger than other relevant time scales, i.e, $t_s >> t_r > T$. Further, t_s is typically about few hundreds of kick cycles and hence we expect this to be experimentally accessible time scale as well. On a much longer time scale as $t \to \infty$, all the energetic particles escape and the steady state decays out.

Indeed, a similar non-equilibrium steady state has been experimentally observed with periodically kicked Bose-Einstein condensate in a finite box for strong kick strengths [5]. These steady states have a classical explanation. Typically, the standard kicked rotor exhibits energy saturation and steady state, for large kick strengths, in the quantum regime due to destructive quantum interferences. [1, 2]. We emphasise that the energy saturation, in our model as well as for the BEC in finite box [5], is induced by the classical effects and leaves a trail in the semiclassical regime.

Fig 6 shows that the quantum mean energy $\langle E \rangle_{in}$ follows the classical curve quite closely. These results cor-

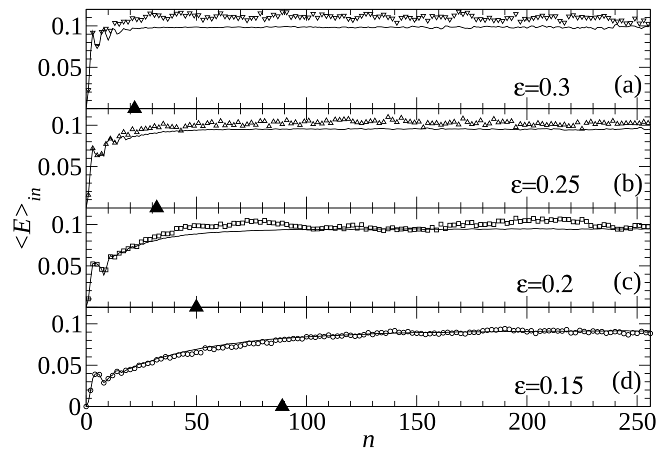

FIG. 6: Non-equilibrium steady state in the system in Hamiltonian Eq. 1. The mean energy for the particles held in between the double barrier structure. The solid lines are the classical results and the symbols correspond to quantum results. The other parameters are $R = 0.5, b = 0.2, \phi = 0, V_0 = 0.5$ and for quantum simulations $\hbar_s = 0.0025$. The solid symbol (triangle up) marks the time scale t_r at which the system relaxes to the steady state.

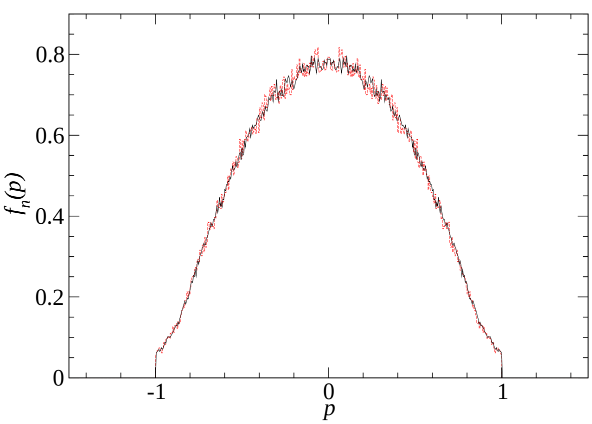

FIG. 7: (Color Online) Classical steady state momentum distribution for $\epsilon = 0.25$ at (a) $n = 100$ (solid) and (b) $n = 200$ (dashed). As seen from Fig. 6(b), the steady state is reached at $n \sim 30$. The other parameters are $R = 0.5, V_0 = 0.5, b = 0.2, \phi = 0$.

respond to $\hbar_s = 0.0025$ and reflect the behaviour in the semiclassical regime. Larger values of ϵ correspond to moving away from semiclassical regime towards purely quantum regime. Thus, we should expect quantum averages to deviate from classical averages in a pronounced manner. This is borne out by the numerical results in Fig 6(a,b,c). There is current interest in quantum nonequilibrium steady states about which not much has been explored until now [21]. For $\epsilon >> 1.0$, the quasiperiodic orbits of the standard map are sufficiently destroyed to allow global transport in phase space. Then, particles do not have to rely on discontinuities in V_{sq} to diffuse in phase space. This leads to unlimited energy absorption by the particles between the barriers and NESS is not supported. Then, the system essentially works like the kicked rotor in the strongly chaotic regime.

B. Energy saturation and steady state

As pointed out earlier, the region covered by the quasiperiodic orbits $C(\mu)$ with $\mu > \mu_c$ is non-diffusive. Now, consider curves $C_\pm(\mu_b)$ such that maximum value of $|p|$ for both these curves is equal to p_c as shown in Fig. 3. Then, all the curves $C(\mu)$ with $\mu < \mu_b$ will have $p \in [-p_c, p_c]$. So any state evolving on one of them will always get reflected at the barriers. Thus, to escape from the finite well, every phase space point in the chaotic region must first reach any $C(\mu)$ with $\mu_b < \mu < \mu_c$. As time $n \to \infty$, all the particles would have escaped from the well and get locked on to one of the invariant curves $C(\mu)$ of the corresponding standard map. Thus, the momenta of escaping particles settle to a stationary distribution on $C(\mu)$ with $\mu_b < \mu < \mu_c$. Thus, the momentum distribution reaches a steady state as $n \to \infty$ and their mean energy $\langle E \rangle$ saturates to $\langle E \rangle_s$.

In Fig. 8(a), the broken curve (blue) shows $\langle E \rangle_s$ for the classical system. As this figure shows, the mean energy of the system increases with time and asymptotically approaches $\langle E \rangle_s$. In the semiclassical regime, we expect a similar behaviour for the quantum average and this is shown as dashed curve in Fig. 8(a). The small difference in saturated values of quantum and classical mean energies can be attributed to the finiteness of Planck's constant which makes its effect felt as ϵ increases.

Further, Fig. 8 (b,c,d) also shows the classical momentum distribution $f_n(p)$ and its quantum analogue $F_n(p) = |\widetilde{\psi}(p,n)|^2$ for the same set of parameters after evolving the system for $n = 250$, 275 and 300 kicking periods. Probability distribution in position representation (not shown here) reveals that at $n = 250, 275$ and 300 the probability density in between the barriers is negligible. Nearly identical distributions in Fig. 8(b,c,d) mark the existence of steady state. Notice that small departures from semiclassical regime is also visible here in the form of slight difference between classical and quantum distributions. For the energy saturation effect, complete chaos between barriers is not essential. If some sticky islands are present between the barriers, the saturated classical and quantum distributions as $n \to \infty$ will display a non-zero component in the finite well region. These non-chaotic components tend to remain localized and will never escape out.

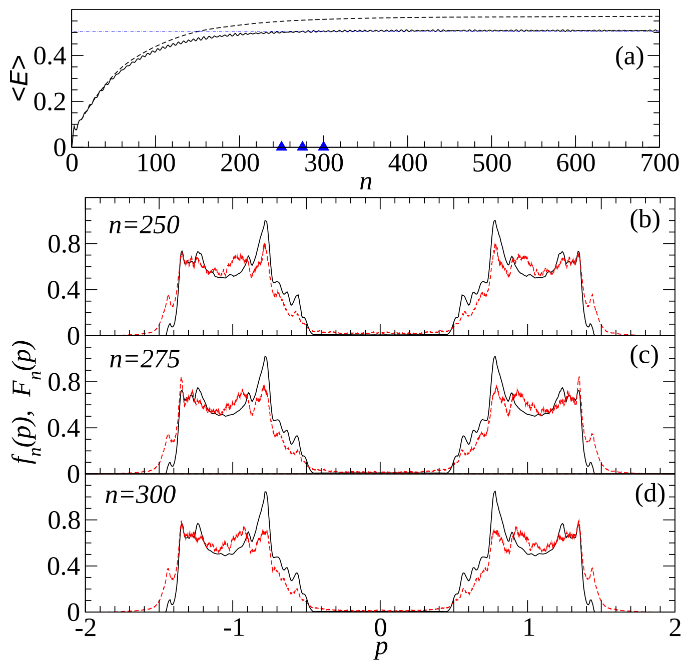

FIG. 8: (Color Online) (a) Classical (solid line) and quantum (dashed line) mean energies as a function of time n for $\epsilon = 0.3$. Other parameters are same as in Fig. 6. Numerically estimated value of $\langle E \rangle_s$ for classical system is shown as a broken line. The triangles in the x-axis are the times for which momentum distribution are shown in (b,c,d). Classical (solid line) and quantum (dashed line) momentum distributions at (b) $n = 250$, (c) $n = 275$ and (d) $n = 300$. Note that the distributions are nearly identical.

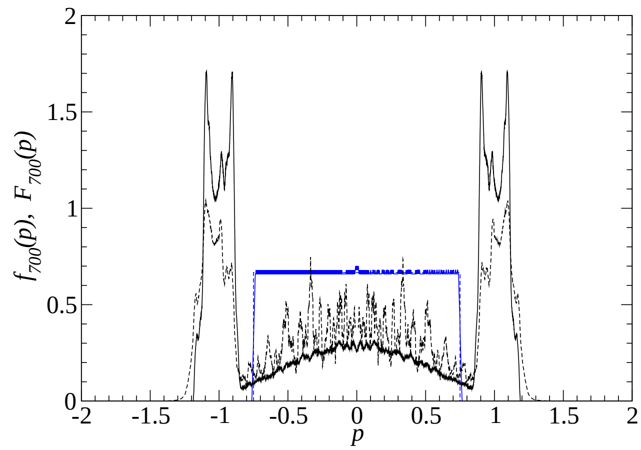

FIG. 9: (Color Online) (a) Classical (solid line) and quantum (dashed line) momentum distributions at $n = 700$ are displayed for $V_0 = 0.5$, $b = 0.2$, $R = 0.5$, $\epsilon = 0.1$. For quantum simulation, $\hbar_s = 0.0025$. The initial distribution at time $n = 0$ is a uniform distribution, the rectangular curve shown in blue. See text for details.

C. Momentum filtering

As demonstrated in section 5(B), when all the chaotic particles exit from the finite well region, a steady state is reached. One possible manifestation of this asymptotic state is the momentum filtering effect that occurs for certain choices of parameters. It is possible to choose system parameters such that momentum distribution of escaped particles becomes narrow. Thus, any broad initial momentum distribution at $n = 0$, after sufficient kicking periods, leads to a distinctly narrow momentum distribution. This is shown in Fig. 9. In this figure, the initial conditions are uniformly distributed in the chaotic layer lying in between the barriers. This chaotic layer also ensures that the final result is independent of the details of the initial distribution. The figure shows the momentum distributions $f_{700}(p)$ (classical) and $F_{700}(p)$ (quantum) plotted for $n = 700$. By this time, a large fraction of particles have escaped from the well and the distribution has become bimodal with distinct peaks near $-p_c$ and p_c. This shows that the double barrier structure, in presence of the kicking field, acts as a momentum filter. We obtain filtering effect for a range of kick-strengths (not shown here) and observe that with decrease in ϵ, the two bands in bimodal distribution become narrow. However, the time at which system approaches steady state corresponding to this bimodal distribution becomes very large. Indeed, since it is experimentally possible to design barrier heights of desired choice, it will be be possible to use double barrier structure to produce filter with desired value of p_c. From Fig. 2, we note that the escaped particles follow extremely close set of invariant curves and their speed, averaged over time, will converge to the winding number of the orbits involved. Hence, the speed distribution will have peaks of infinitesimal width at $|p_c|$. It is pertinent to note that a momentum filtering effect based on a very different mechanism has been studied by Monteiro et. al. in the context of a variant of kicked rotor model [22].

We emphasise that all the dynamical features discussed in section 5(A,B,C) can be explained on the basis of (i) co-existence of diffusive and non-diffusive regions which exists because the non-KAM nature of the system affects the dynamics differently in different phase space regions, and (ii) presence of KAM curves through out the phase space outside the double barrier region. Hence, all the dynamical features can be attributed to the interplay between the KAM and non-KAM behaviour of the system.

VI. DISCUSSIONS AND SUMMARY

In summary, we have presented primarily numerical results of the dynamics of non-interacting particles in a double barrier structure acted upon by periodic kicking field. This model differs from the paradigmatic kicked rotor. This is essentially a non-KAM system and hence chaotic dynamics sets in for even for infinitesimal excursions from the integrable limit of kicking strength $\epsilon = 0$. Further, this displays non-equilibrium steady state and

classically induced suppression of energy growth in the semiclassical regime. This is in contrast with the classical kicked rotor that displays diffusion only for $\epsilon >> 1$ and its quantum version arrests this through dynamical localisation, an outcome of quantum interferences.

Some of the earlier works on the double barrier type potential have considered it as a scattering problem, in a different setting with a drive term. For example, see references [23]. An incoming wavepacket hits the left barrier (see Fig. 1) and tunnels in to it and, depending upon the parameters chosen, some or all of it emerges out of the right barrier. This mechanism requires purely quantum effects such as tunneling and in this work we have deliberately avoided them to focus on the semiclassical regime. Since tunneling probability is nearly zero in this semiclassical setting, any initial distribution placed anywhere outside the barriers ($|x| > a + b$) will continue to evolve on the KAM like invariant tori. However, based on the results obtained in this paper, we can speculate about the case when quantum effects come into play. Tunneling will allow a wavepacket to enter the through the left barrier and non-KAM chaos will ensure that it gets dispersed. But now, the wavepacket can tunnel out through the right barrier. This scenario could potentially lead to an interesting competition between above barrier crossings and tunneling. Another interesting case relates to periodic version of this model which can also be used for directed transport. We are pursuing these questions and will be reported elsewhere.

The dynamical features in our model such as the non-equilibrium steady state and classically induced energy growth suppression are of current interest in the general context of transport and localisation especially for interacting systems such as the Bose-Einstein condensates. Recently there have been several experimental results that point to classical features suppressing energy growth of condensates [14]. Typically, in such experiments, condensates are released from a confining potential and their expansion in a disordered potential is studied. When chemical potential $\mu < V_0$, where V_0 is the strength of disorder, condensates are classically reflected from the fluctuations of the disordered potential effectively localising the condensates. In our model, particles are neither interacting nor there is any disordered potential. However, the non-KAM chaotic dynamics and KAM like invariant curves provide the essential ingredient for the suppression of diffusion. Even as the particles are transported in the position space their energy absorption is restricted as $t \to \infty$ by KAM like structures. Such studies form an important background to understand and clearly distinguish similar quantum phenomena like the Anderson localisation from the classically induced ones and also to explore the connections between interactions, localisation and disorder.

Quantum chaos in double barrier potentials have been studied before experimentally using GaAs/AlGaAs heterostructures [24] though not with a periodic kicking field. In these experiments electrons tunnel through the double barrier potential and chaos is induced within the barriers due to the field created by the charge accumulation in the well [24]. Since resonant tunnelling plays an important role in this experiment, this can be regarded as being quantum in nature without classical analog. The double barrier system in Eq. 1 could be used with resonant tunnelling to study purely quantum effects as well though in the present work we have primarily explored the classical and semiclassical features. The foregoing arguments also imply that the system can also be realized experimentally in a laboratory. The cold atoms in optical lattices is the testing ground for variants of kicked rotor. An experimental set-up involving cold atoms, optical lattices with double barrier heterostructures should be possible.

Currently there is considerable interest in the exciting field of chaotic ratchets [6]. Generally, ratchets are systems with broken spatio-temporal symmetries from which directed transport can be obtained even in the absence of a net bias. There have been several proposals and at least one experimental realization for a chaotic ratchet in the last few years. The system presented in this work lacks the spatial periodicity required of a ratchet. But the kicking potential, being sinusoidal, is already spatially periodic. Further, from a theoretical perspective, it is not difficult to have spatially periodic double barrier structures. Then, it might become possible to realize ratchet dynamics in this system. All the existing chaotic ratchet proposals are based on systems that obey KAM theorem. The model presented in this work might lead to new ways to use non-KAM type dynamics for deterministic, directed transport.

Acknowledgments

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APPENDIX A

Consider a particle that evolves on an invariant curve of the standard map $C_5(\mu_5)$, approaches right barrier at $x_w = R\pi$ with $p > p_c$ during its motion after $n^{th}-$kick, crosses it and exits on to another invariant curve of standard map $C_6(\mu_6)$. In this appendix, we show that as the width of the barrier $b \to 0$, $C_5(\mu_5) \to C_6(\mu_6)$.

After the particle crosses the interface at x_w and if Δt denotes the time it will take to cross the barrier region of width b, then $\Delta t \to 0$ if $b \to 0$. Hence, the probability that a particle will experience the next kick while crossing the barrier will also tend to zero. Hence we can assume

that the particle does not experience a kick while crossing the barrier. In such a situation, the particle will face only two discontinuities between nth and $(n+1)$th kick. Thus, $k=2$, $B_1 = x_w$ and $B_2 = x_w + b$. From our assumptions, $\begin{pmatrix} x_n^0 \\ p_n^0 \end{pmatrix}$ lie on $C_5\,(\mu_5)$, and $\begin{pmatrix} x_n^2 \\ p_n^2 \end{pmatrix}$ will lie on $C_6\,(\mu_6)$.

$$\begin{pmatrix} x_n^1 \\ p_n^1 \end{pmatrix} = \widehat{\mathcal{R}}_1 \begin{pmatrix} x_n^0 \\ p_n^0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_w + \dfrac{(x_n^0 - x_w)\, p_n^1}{p_n^0} \\ \sqrt{(p_n^0)^2 - 2V_0} \end{pmatrix} \tag{A1}$$

Similarly,

$$\begin{pmatrix} x_n^2 \\ p_n^2 \end{pmatrix} = \widehat{\mathcal{R}}_2 \begin{pmatrix} x_n^1 \\ p_n^1 \end{pmatrix} \Rightarrow \begin{pmatrix} x_w + b + \dfrac{(x_n^1 - x_w - b)\, p_n^2}{p_n^1} \\ \sqrt{{p_n^1}^2 - 2V_0} \end{pmatrix}. \tag{A2}$$

Substituting for x_1 and p_1 from Eq. A1 in Eq. A2, we get,

$$\begin{pmatrix} x_n^2 \\ p_n^2 \end{pmatrix} = \begin{pmatrix} b - \dfrac{bp_n^0}{p_n^1} + x_n^0 \\ p_n^0 \end{pmatrix} \tag{A3}$$

Using $b \to 0$, we get, $\begin{pmatrix} x_n^2 \\ p_n^2 \end{pmatrix} \to \begin{pmatrix} x_n^0 \\ p_n^0 \end{pmatrix}$. This implies $C_5(\mu_5) \to C_6(\mu_6)$ or $\mu_6 - \mu_5 \to 0$.

APPENDIX B

We show that for certain special choices of (R, ϕ), reflection from the walls of potential V_{sq} takes a state from invariant curve C_+ to its symmetric counterpart C_-, where C_+ and C_- are related through reflection symmetry about $(0,0)$. Let

$$\left\{ \begin{array}{c} R\pi + \phi = l\pi \\ -R\pi + \phi = m\pi \end{array} \right\}, \quad l, m \in \text{integer} \tag{B1}$$

Then, $x_r = l\pi$ and $-x_l = m\pi$. Let $\begin{pmatrix} x_n^{i-1} \\ p_n^{i-1} \end{pmatrix}$ lie on C_+. Reflection from the right boundary at x_r will take it to

$$\begin{pmatrix} x_n^i \\ p_n^i \end{pmatrix} = \widehat{\mathcal{R}}_i \begin{pmatrix} x_n^{i-1} \\ p_n^{i-1} \end{pmatrix} = \begin{pmatrix} 2l\pi - x_n^{i-1} \\ -p_n^{i-1} \end{pmatrix} \tag{B2}$$

on the invariant curve C. The spatial periodicity of 2π in the standard map implies that

$$\begin{pmatrix} (2l\pi - x_n^{i-1}) \bmod (2\pi) \\ -p_n^{i-1} \end{pmatrix} = \begin{pmatrix} -x_n^{i-1} \\ -p_n^{i-1} \end{pmatrix} \tag{B3}$$

is on C. Since $\begin{pmatrix} -x_n^{i-1} \\ -p_n^{i-1} \end{pmatrix}$ is on C_- and C_- is unique, we have $C = C_-$. Thus, the effect of reflection from the right boundary at x_r is to take a state from C_+ to C_- if Eq. B1 is satisfied. Similarly, the effect of reflection from left boundary at $-x_l$ is to take a state from C_- to C_+.

[1] B. V. Chirikov, Phys. Rep. **52**, 263 (1979); F. M. Izrailev, Phys. Rep. **196**, 299 (1990); G. Casati in *Quantum Chaos : Between Order and Disorder*, (Cambridge University Press, Cambridge, 1995).

[2] B. V. Chirikov in *Chaos and quantum physics*, M. -J. Giannoni, A. Voros and J. Zinn-Justin (eds), (North-Holland, 1991).

[3] L. E. Reichl, *The transition to chaos : Conservative classical systems and quantum manifestations*, (Springer, New York, 2004).

[4] G. Casati, B. V. Chirikov, J. Ford and F. M. Izrailev, Lect. Notes Phys. **93**, 334 (1979); S. Fishman, D. R. Grempel and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).

[5] F. L. Moore et. al., Phys. Rev. Lett. **73**, 2974 (1994); F. L. Moore *et. al.*, Phys. Rev. Lett. **75**, 4598 (1995).

[6] T. Salger *et. al.*, Science **326**, 1241 (2009); A. Kenfack, J. Gong, and A. K. Pattanayak , Phys. Rev. Lett. **100**, 044104 (2008); S. Flach *et. al.*, Phys. Rev. Lett. **84**, 2358 (2000); H. Schanz *et. al.*, Phys. Rev. Lett. **87**, 070601 (2001). T. S. Monteiro *et. al.*, Phys. Rev. Lett. **89**, 194102 (2002); E. Lundh and M. Willin, Phys. Rev. Lett. **94**, 110603 (2005); J. Gong and P. Brumer, Phys. Rev. Lett. **97**, 240602 (2006).

[7] G. M. Zaslavsky, *Hamiltonian chaos and fractional dynamics*, (Oxford University Press, UK, 2005).

[8] R. Sankaranarayanan, A. Lakshminarayan and V. B. Sheorey, Phys. Rev. E **64**, 046210 (2001); Phys. Lett. A **279**, 313 (2001).

[9] K. Takahashi and N. Saito, Phys. Rev. Lett. **55**, 645 (1985).

[10] B. Hu *et. al.*, Phys. Rev. Lett. **82**, 4224 (1999).

[11] T. M. Fromhold *et. al.*, Nature **428**, 726 (2004); T. M. Fromhold *et. al.*, Phys. Rev. Lett. **87**, 046803 (2001).

[12] S. A. Gardiner *et. al.*, Phys. Rev. Lett. **79**, 4790 (1997).

[13] K. Henderson et. al., Europhys. Lett. **75**, 392 (2006).

[14] D. Clement *et. al.*, Phys. Rev. Lett. **95**, 170409 (2005); C. Fort *et. al.*, Phys. Rev. Lett. **95**, 170410 (2005); T. Schulte *et. al.*, Phys. Rev. Lett. **95**, 170411 (2005); L. Sanchez-Palencia *et. al.*, New J. Phys. **10**, 045019 (2008).

[15] T. P. Meyrath et. al., Phys. Rev. A **71**, 041604(R) (2005).

[16] D. Poletti et. al., Phys. Rev. E **73**, 056203 (2006).

[17] B. Galdrikian, B. Birnir and M. Sherwin, Phys. Lett. A **203**, 319 (1995).
[18] Paul Harrison, *Quantum Wells, Wires and Dots*, (John Wiley and Sons, West Sussex, 2005).
[19] D. J. Tannor, *Introduction to Quantum Mechanics : A time dependent perspective*, (University Science Books, Sausalito, 2007).
[20] “Fastest Fourier Transform in the West”, `www.fftw.org`
[21] M. Znidaric, J. Stat. Mech. L05002 (2010).
[22] T. Jonckheree, M. Isherwood and T. S. Monteiro, Phys. Rev. Lett **91** 253003 (2003).
[23] M. Garttner *et. al.*, Phys. Rev. E **81**, 051136 (2010); M. Wagner, Phys. Rev. A **51**, 798 (1995); I. Vorobeichik, R. Lefebvre and N. Moiseyev, EPL **41**, 111 (1998); M. Wagner, Phys. Rev. B **49**, 16544 (1994); For two barrier quantum pumps, see M. M. Mahmoodian, L. S. Braginsky and M. V. Entin, Phys. Rev. B **74**, 125317 (2006).
[24] G. Jona-Lasinio, C. Presilla and F. Capasso, Phys. Rev. Lett. **68**, 2269 (1992); C. Presilla, G. Jona-Lasinio and F. Capasso, Phys. Rev. B **43**, 5200(R) (1991); A. J. McNary and Ashok Puri, J. Appl. Phys. **80**, 247 (1996); A. Rosa, A. Puri, J. E. Murphy and T. Odagaki, J. Appl. Phys. **75**, 5196 (1994).